J. Fluid Mech. (1965), *vol.* 21, *part* 3, *pp.* 453–464
Prtnted in Great Britain

Laminar convection in a heated vertical tube rotating about a parallel axis

By W. D. MORRIS
Department of Mechanical Engineering, University of Liverpool

(Received 3 January 1964 and in revised form 30 July 1964)

This paper presents an investigation into the influence of rotation on the laminar asymptotic velocity and temperature profiles obtained when fluid flows through a vertical tube which rotates about a parallel axis with uniform angular velocity, and which is subjected to a uniform temperature gradient. Rotation induces secondary free convection flow in the plane perpendicular to the axis resulting in non-symmetrical axial-velocity and temperature profiles which modify the resistance to flow and rate of heat transfer. The conservation equations are solved using a series expansion in ascending powers of the rotational Rayleigh number, the resulting solutions being approximate and valid only for low rates of heating.

1. Introduction

To enable the rotating components of certain machines to operate continuously while located within a high temperature environment it is often necessary to employ some method of cooling. For example, an improvement in the thermal efficiency and power output of a gas turbine may be achieved if the temperature of the gas entering the turbine is increased. Unfortunately, present-day materials for rotor blades cannot operate successfully at temperatures in excess of 1500 F and so, if gas temperatures above this value are envisaged, some method of rotor blade cooling is essential. Holzworth (1938) appears to be one of the first people to suggest that this problem could be solved by the use of blind radial holes situated in the rotor blade and filled with a suitable fluid. Owing to the intense centripetal acceleration present, free convection currents occur in the fluid causing the warmer fluid to move towards the axis of rotation. This stream of warm fluid adjacent to the wall is simultaneously replaced by a central core of relatively cool fluid located in the main rotor shaft, resulting in the blade material being maintained at a temperature level compatible with its mechanical strength. Theoretical analyses of the flow process and heat transfer inside these thermosyphon holes have been carried out by Lighthill (1953) and Leslie (1960), and these have largely been verified by the experimental work of Martin & Cohen (1954) and Martin (1955).

The motion of the fluid in the thermosyphon tubes described above is entirely due to free convection. In many non-rotating heat transfer systems the influence of free convection (due to the earth's gravitation) is often neglected in comparison with pressure gradients. However, it is interesting to note that in a gas turbine, where the centripetal acceleration may be as high as $10^4 g$, the free convection velocities encountered are as high as those usually associated with forced convection in non-rotating devices.

Cooling of turbine rotor blades has also been effected by the forced circulation of suitable coolant through internal passages (Alpert, Grey & Flashar 1960), the circulation of the fluid being maintained by externally generated pressure gradients. Owing to the radial and tangential components of acceleration caused by rotation, modifications to this otherwise forced convection process occur. This class of problem, where forced cooling of rotating components occurs, is not only restricted to the field of gas turbines. The power output from electrical machines is to some extent governed by the permissible temperature rise in the insulation surrounding the rotor conductors. Although cooling of these conductors is commonly achieved by the forced circulation of air over the rotor periphery, there are obvious advantages to be gained if the heat transfer is effected to a suitable coolant flowing inside the conductors themselves.

It is thus evident that the problem of forced flow through heated rotating channels is interesting both academically and practically. The present paper is confined to an analysis of one of many possible configurations. This configuration, illustrated in figure 1, takes the form of a vertical cylindrical tube rotating about an axis parallel to itself with uniform angular velocity. While the tube rotates, fluid is pumped through in an upward direction and the tube wall is subjected to a uniform temperature gradient. Although the flow in the entry region is of significant interest, the present analysis is confined to distances along the tube which are larger than the hydrodynamic and thermal entry lengths.

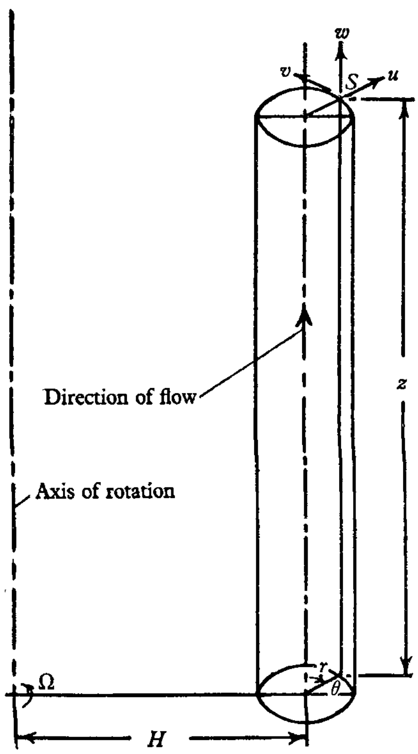

FIGURE 1. Physical model and co-ordinates.

2. Formulation of the problem

The flow is assumed laminar and, with the exception of density, the fluid properties are taken to be constant. If the velocity components at a point S in the co-ordinate directions shown in figure 1 are u, v, and w, then since these velocities refer to a moving frame of reference, the respective acceleration components at S are given by

$$f_r = u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}-2\Omega v-(r+H\cos\theta)\,\Omega^2, \tag{1}$$

$$f_\theta = u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+2\Omega u+H\Omega^2\sin\theta, \tag{2}$$

$$f_z = u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}, \tag{3}$$

where Ω is the angular velocity of the tube and H is the distance between the axis of rotation and the tube axis. With the exception of terms involving Ω, the acceleration components above are the same as those in a non-rotating tube. The additional terms may be thought of as body forces, similar to gravity, producing free convection currents if fluid temperature gradients are present. In order to include the effect of free convection in the basic conservation laws density must vary with temperature. This variation is sufficiently small, however, to be ignored in all terms except the buoyancy force in the momentum equations.

For the uniformly heated tube considered, if the wall material is of sufficiently high thermal conductivity to ignore any circumferential temperature variations, the tube wall temperature T_W may be expressed as

$$T_W = T_0+\tau z, \tag{4}$$

where T_0 is the wall temperature at the origin and τ is the axial temperature gradient. For distances along the tube large enough to permit entry effects to be ignored, the continuity equation may be satisfied by specifying a dimensionless stream function ψ as

$$\nu(\partial\psi/\partial r) = -v \quad \text{and} \quad \nu(\partial\psi/\partial\theta) = ru, \tag{5}$$

where ν is the kinematic viscosity. Because distances well away from inlet influences are being considered, the pressure distribution must be of form

$$p = \gamma z+p(r,\theta), \tag{6}$$

where γ is the axial pressure gradient and $p(r,\theta)$ a function specifying the distribution of pressure in the (r,θ)-plane.

For this fully developed state the equations expressing conservation of momentum with negligible viscous dissipation may be written in the non-dimensional form

$$\nabla^4\psi+\frac{1}{x}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,\theta)}+A\left(\frac{1}{x}\frac{\partial\eta}{\partial\theta}\cos\theta+\frac{\partial\eta}{\partial x}\sin\theta\right)+A\epsilon\frac{\partial\eta}{\partial\theta}+\frac{AP}{R}\frac{\partial(\eta,\psi)}{\partial(x,\theta)} = 0, \tag{7}$$

$$\nabla^2W+\frac{1}{x}\frac{\partial(\psi,W)}{\partial(x,\theta)}+4R-\eta = 0, \tag{8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}, \quad x = r/a, \quad \eta = (T_w - T)/\tau\sigma a, \quad W = aw/\nu, \quad \epsilon = a/H,$$

and $\sigma = \nu/\alpha$ the Prandtl number,

$R = \dfrac{-a^3}{4\nu^2\rho}\dfrac{\partial p}{\partial z}$ the Reynolds number,

$A = \dfrac{\beta H\Omega^2\tau a^4}{\alpha\nu}$ the rotational Rayleigh number,

$B = \dfrac{\beta g\tau a^4}{a\nu}$ the gravitational Rayleigh number,

and $S = \dfrac{-a^2}{2H\Omega\rho\nu}\dfrac{\partial p}{\partial z}$ the Rossby number;

a is the tube radius, α the thermal diffusivity, β the coefficient of expansion, ρ the density and T the local temperature of the fluid.

In (7) and (8) the buoyancy forces have been calculated relative to the tube wall temperature with the remaining force distribution absorbed into the pressure terms. Equation (7) results from the cross-differentiation of the momentum equations in the r and θ directions, thus eliminating the pressure terms.

Since laminar motion of the fluid is considered, heat transfer between adjacent layers is due to molecular conduction. The energy equation in dimensionless form thus reduces to

$$\nabla^2\eta + \frac{\sigma}{x}\frac{\partial(\psi,\eta)}{\partial(x,\theta)} + W = 0. \tag{9}$$

Solutions of (7), (8) and (9) are subject to the following boundary conditions. At the tube wall $u = v = w = \eta = 0$, and at the tube axis u, v, w, and η must remain finite.

3. Solution of the equations

Although an exact solution of (7), (8) and (9) would be extremely difficult to find, if indeed possible, an approximate solution may readily be obtained using a technique suggested by Lighthill (1949). Applied to the present problem the technique involves the expansion of the non-dimensional velocity and temperature fields in ascending powers of a suitable small parameter. It is necessary that the parameter selected be small in magnitude for the solutions obtained to be valid.

The technique has been successfully used by Barua (1954) for the determination of secondary flows due to rotation in an unheated tube. In this case the tube was considered to rotate with uniform angular velocity about an axis perpendicular to it. Morton (1959) also used the technique to study the influence of gravitational free convection in a uniformly heated horizontal tube with forced laminar flow. The analysis presented in this paper is similar to that used by Morton. However, owing to the influence of tangential as well as radial acceleration components in the present problem, the resulting equations are more complicated.

In order to solve (7), (8) and (9), ψ, W, and η are expanded in terms of the rotational Rayleigh number, the parameter governing the free convection due to rotation. Thus we consider

$$\psi = \psi_0 + A\psi_1 + A^2\psi_2 + \ldots, \tag{10}$$

$$W = W_0 + AW_1 + A^2W_2 + \ldots, \tag{11}$$

$$\eta = \eta_0 + A\eta_1 + A^2\eta_2 + \ldots. \tag{12}$$

On substitution of (10)–(12) into (7)–(9) sets of equations for the zeroth-, first- and second-order coefficients are obtained by equating powers of the rotational Rayleigh number. Since there can be no flow in the (r, θ)-plane when $A = 0$, it follows that $\psi_0 = 0$. The resulting equations for coefficients up to and including second order are as follows.

Zeroth-order:

$$\nabla^2 W_0 = -4R, \tag{13}$$

$$\nabla^2 \eta_0 = -W_0. \tag{14}$$

First-order:

$$\nabla^4 \psi_1 = -\left[\frac{1}{x}\frac{\partial \eta_0}{\partial \theta}\cos\theta + \frac{\partial \eta_0}{\partial x}\sin\theta\right] - \epsilon\frac{\partial \eta_0}{\partial \theta}, \tag{15}$$

$$\nabla^2 W_1 = \frac{1}{x}\frac{\partial(W_0, \psi_1)}{\partial(x, \theta)} + \frac{B}{A}\eta_0, \tag{16}$$

$$\nabla^2 \eta_1 = \frac{\sigma}{x}\frac{\partial(\eta_0, \psi_1)}{\partial(x, \theta)} - W_1. \tag{17}$$

Second-order:

$$\nabla^4 \psi_2 = \frac{1}{x}\frac{\partial(\nabla^2\psi_1, \psi_1)}{\partial(x, \theta)} - \left[\frac{1}{x}\frac{\partial \eta_1}{\partial \theta}\cos\theta + \frac{\partial \eta_1}{\partial x}\sin\theta\right] - \epsilon\frac{\partial \eta_1}{\partial \theta} + \frac{P}{R}\frac{\partial(\psi_1, \eta_0)}{\partial(x, \theta)}, \tag{18}$$

$$\nabla^2 W_2 = \frac{1}{x}\frac{\partial(W_1, \psi_1)}{\partial(x, \theta)} + \frac{1}{x}\frac{\partial(W_0, \psi_2)}{\partial(x, \theta)} + \frac{B}{A}\eta_1, \tag{19}$$

$$\nabla^2 \eta_2 = \frac{\sigma}{x}\frac{\partial(\eta_1, \psi_1)}{\partial(x, \theta)} + \frac{\sigma}{x}\frac{\partial(\eta_0, \psi_2)}{\partial(x, \theta)} - W_2. \tag{20}$$

In (16) and (19) B/A has been used for the acceleration ratio $g/H\Omega^2$.

Solution of (13)–(20) is not in itself difficult. However, for solutions of higher order the amount of labour required is considerable. Solutions up to second order are as listed below.

Zeroth-order solutions

$$W_0 = R(1 - x^2), \tag{21}$$

$$\eta_0 = \tfrac{1}{16}R(1 - x^2)(3 - x^2). \tag{22}$$

These expressions are those obtained by Nusselt for forced laminar convection in a uniformly heated tube, reported in Goldstein (1957). No account of rotation has been taken at this stage.

First-order solutions

$$\psi_1 = \frac{R}{4608} x(1-x^2)(10-x^2)\sin\theta, \tag{23}$$

$$W_1 = \frac{R^2 x}{1{\cdot}843\times 10^5}(1-x^2)(49-51x^2+19x^4-x^6)\cos\theta + \frac{BR}{576A}(x^6-9x^4+27x^2-19), \tag{24}$$

$$\eta_1 = \frac{R^2}{2{\cdot}212\times 10^7}[(381+1325\sigma)x-(735+3000\sigma)x^3 + (500+2600\sigma)x^5-(175+1125\sigma)x^7 + (30+210\sigma)x^9-(1+10\sigma)x^{11}]\cos\theta - \frac{BR}{3{\cdot}686\times 10^4 A}(211-304x^2+108x^4-16x^6+x^8). \tag{25}$$

Second-order solutions

Because the second-order solutions contain numerical coefficients of an unwieldy nature, they have been grouped within summation signs and the actual values tabulated in the Appendix.

$$\psi_2 = \frac{R^2\sin 2\theta}{2{\cdot}123\times 10^7}\left[\sum_{b=1}^{7}(C_{2b}+D_{2b}x^{2b})\right] + \frac{R^2\epsilon\sin\theta}{2{\cdot}212\times 10^7}\left[\sum_{c=0}^{7}(C_{2c+1}+D_{2c+1}\sigma)x^{2c+1}\right] + \frac{S\cos\theta}{1{\cdot}843\times 10^4}\left[\sum_{d=0}^{6}(C_{2d+1}x^{2d+1})\right] - \frac{BR\sin\theta}{4{\cdot}424\times 10^7 A}\left[\sum_{e=0}^{5}(C_{2e+1}x^{2e+1})\right], \tag{26}$$

$$W_2 = \frac{R^3}{(4608)^2}\left[\sum_{b=0}^{8}(E_b x^b)\right] - \frac{R^3\cos\theta}{2{\cdot}123\times 10^7}\left[\sum_{c=1}^{8}(E_{2c}+F_{2c}\sigma)x^{2c}\right] + \frac{BR^2\cos\theta}{2{\cdot}123\times 10^7 A}\left[\sum_{d=0}^{6}(E_{2d+1}+F_{2d+1}\sigma)x^{2d+1}\right] - \frac{R^3\epsilon\cos\theta}{1{\cdot}106\times 10^7}\left[\sum_{e=0}^{8}(E_{2e+1}+F_{2e+1}\sigma)x^{2e+1}\right] + \frac{R^2 S\sin\theta}{9216}\left[\sum_{i=0}^{7}(E_{2i}x^{2i+1})\right] - \frac{B^2R}{3{\cdot}686\times 10^4 A^2}\left[\sum_{j=0}^{5}(E_{2j}x^{2j})\right], \tag{27}$$

$$\eta_2 = \frac{R^3}{2{\cdot}123\times 10^7}\left[\sum_{b=0}^{9}(J_{2b}+K_{2b}\sigma+L_{2b}\sigma^2)x^{2b}\right] + \frac{R^3\cos 2\theta}{2{\cdot}123\times 10^7}\left[\sum_{c=0}^{9}(J_{2c}+K_{2c}\sigma+L_{2c}\sigma^2)x^{2c}\right] + \frac{B^2R^2\cos\theta}{2{\cdot}123\times 10^7 A^2}\left[\sum_{d=0}^{7}(J_{2d+1}+K_{2d+1}\sigma)x^{2d+1}\right] - \frac{R^3\epsilon\cos\theta}{2{\cdot}123\times 10^7}\left[\sum_{e=0}^{9}(J_{2e+1}+K_{2e+1}\sigma+L_{2e+1}\sigma^2)x^{2e+1}\right] + \frac{R^2 S\sin\theta}{7{\cdot}373\times 10^4}\left[\sum_{i=0}^{8}(J_{2i+1}+K_{2i+1}\sigma)x^{2i+1}\right] + \frac{B^2R}{3{\cdot}686\times 10^4 A^2}\left[\sum_{j=0}^{6}(J_{2j}x^{2j})\right]. \tag{28}$$

4. Effect of rotation on flow resistance and heat transfer

The pressure loss along a tube, due to viscous shear, is proportional to the gradient of the axial-velocity profile at the boundary. Since the axial-velocity profile has been shown to deviate from the parabolic form associated with laminar flow in a non-rotating tube, there must be a consequential variation in the resistance to flow.

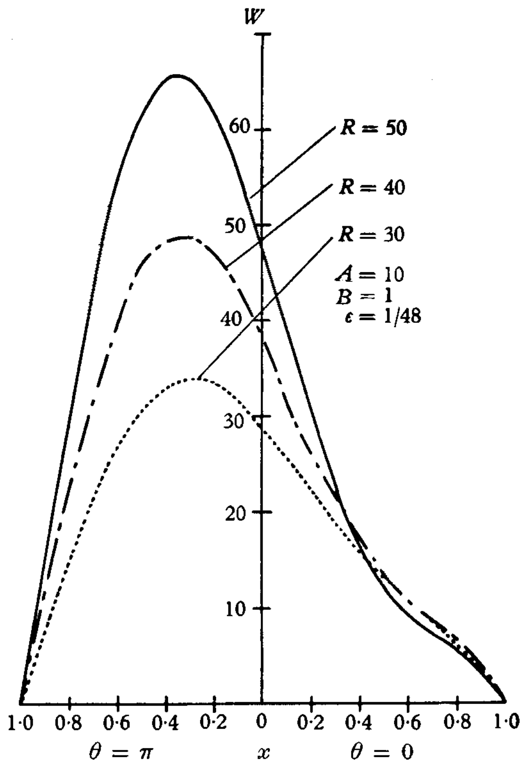

FIGURE 2. Non-dimensional axial velocity distribution across tube diameter $\theta = 0$. Fluid: water at 100 °F.

Typical non-dimensional axial-velocity profiles across a tube diameter $\theta = 0$ are shown in figure 2. These curves have been evaluated for $\epsilon = \frac{1}{48}$, the fluid considered being water at 100 °F. The curves are drawn for a range of Reynolds number values, with fixed values of the rotational Rayleigh number and acceleration ratio. These values are listed on the curves. For similar conditions the non-dimensional temperature distributions are shown in figure 3.

Under the influence of the radial component of acceleration the cooler and thus less dense particles of fluid tend to move away from the axis of rotation causing the portion of the tube furthest away from the axis of rotation to be in contact with the relatively cooler fluid.

The viscous shear experienced by a fluid moving in a conduit is usually expressed in terms of a resistance coefficient C_f defined by

$$C_f = (-a/w_m^2\rho)\,\partial p/\partial z, \tag{29}$$

where w_m is the dimensional mean velocity. The solutions given above result in the following expression for w_m

$$w_m = 2R/a[\tfrac{1}{4} - 0{\cdot}0525(RA/4608)^2 - 0{\cdot}0072B(1-0{\cdot}0299B)]. \tag{30}$$

Hence

$$C_f = \frac{16}{R[1-0{\cdot}2100(RA/4608)^2 - 0{\cdot}0288B(1-0{\cdot}0299B)]^2}. \tag{31}$$

For zero rotation and isothermal flow (31) reduces to $C_f = 16/R$, which is the well-known value for the laminar resistance coefficient in a non-rotating tube.

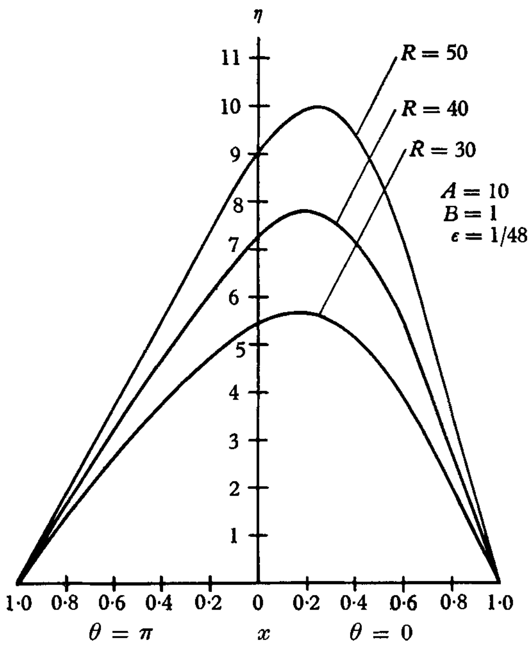

FIGURE 3. Non-dimensional temperature distribution across tube diameter $\theta = 0$. Fluid: water at 100 °F.

A further consequence of rotation is a modification of the rate of heat transfer from the tube wall to fluid. This is by virtue of the non-symmetrical temperature distribution formed, an example of which is illustrated in figure 3. The heat transfer rate $\dot{q}$ across the solid–liquid interface may be written in terms of the non-dimensional temperature as

$$\dot{q} = ka\tau\sigma \int_0^{2\pi} (\partial\eta/\partial x)_{x=1}\, d\theta, \tag{32}$$

where k is the thermal conductivity of the fluid. It is usual to express a heat-transfer rate in terms of a dimensionless Nusselt number N, which for this problem may be defined by

$$N = \dot{q}/\pi k(T_W - T_m), \tag{33}$$

where T_m is the mean fluid temperature across the tube section,

$$T_m = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} (T_W - \tau a\sigma\eta)\, x\, dx\, d\theta. \tag{34}$$

Evaluation of (32) and (34) yields the following expression for N on substitution into (33),

$$N = \frac{[a^1 - b^1B + c^1B^2 - (RA/d^1)^2(e^1 + f^1\sigma + g^1\sigma^2)]}{[j^1 - B/k^1(l^1 - m^1B) - (RA/d^1)^2(n^1 + s^1\sigma + t^1\sigma^2)]}, \tag{35}$$

where

$$\begin{aligned} a^1 &= 0{\cdot}2500, & j^1 &= 0{\cdot}0417,\\ b^1 &= 0{\cdot}0072, & k^1 &= 3{\cdot}686\times 10^4,\\ c^1 &= 0{\cdot}0002, & l^1 &= 45{\cdot}60,\\ d^1 &= 4608, & m^1 &= 1{\cdot}3631,\\ e^1 &= 0{\cdot}0328, & n^1 &= 0{\cdot}0133,\\ f^1 &= 0{\cdot}0000, & s^1 &= 0{\cdot}0035,\\ g^1 &= 0{\cdot}0018, & t^1 &= 0{\cdot}0009. \end{aligned}$$

With no rotation of the tube (35) reduces to $N = 6$ if gravitational buoyancy is ignored. This is the result obtained by Nusselt, reported by Goldstein (1957), for forced laminar convection in cylindrical tubes.

5. Discussion

Solutions have been obtained for the velocity and temperature distributions for fluid flowing in laminar motion in a uniformly heated vertical tube, when the tube is rotating about an axis parallel to itself with uniform angular velocity. Owing to the complicated nature of the equations specifying the flow conditions, the solutions presented are approximate, being valid only for low rates of heating. However, the results do give some indication of the manner in which rotation affects the resistance to flow and heat-transfer rate.

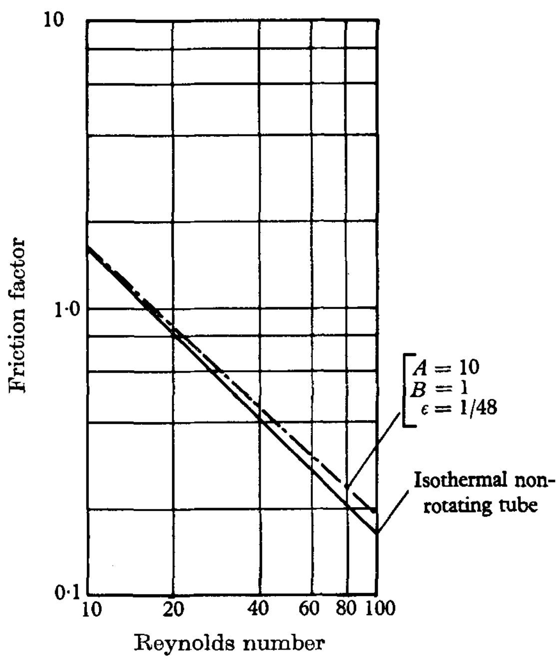

FIGURE 4. Typical variation of friction factor with Reynolds number.

Figure 4 illustrates the variation of resistance coefficient with Reynolds number for values of the rotational Rayleigh number and acceleration ratio shown. These curves were evaluated for similar geometric and flow conditions as the non-dimensional velocity and temperature profiles shown in figures 2 and 3. From figure 4 it is seen that at a Reynolds number of 100 the resistance coefficient is increased by approximately 12·5 % above the value for a non-rotating tube, even though a low rate of heat transfer has been considered. Similarly, evaluations of the Nusselt number for the same range of parameters indicates an increase of approximately 5 % at a Reynolds number of 100. It is seen, thus, that percentage increase in resistance is greater than the corresponding increase in heat transfer.

In this paper, where laminar convection in a rotating vertical tube has been considered, three components of convection may be identified. Circulation of the fluid through the tube is maintained by an externally generated pressure gradient so that forced convection results in a constant value of 6 for the Nusselt number, if other convective effects are omitted. Owing to gravitational buoyancy in the axial direction this Nusselt number value is modified, the magnitude of the alteration being characterized by the gravitational Rayleigh number B.

Finally, rotational buoyancy due to centrifugal and Coriolis forces causes convective currents across a tube section. The centrifugal buoyancy in (35) is seen to depend on the product of the rotational Rayleigh number and the Reynolds number, whereas gravitational buoyancy is dependent upon the gravitational Rayleigh number alone. This feature is analogous to the non-rotating horizontal tube studied by Morton (1959). In his work on planetary waves Rossby (1949) showed that the ratio of inertia force to Coriolis force was an important parameter in rotating flows. This ratio, known as the Rossby number, S, thus enables a measure of the importance of the Coriolis buoyancy to be made. Although the Rossby number occurs in the velocity and temperature fields for solutions up to second order it has no ultimate influence on the heat transfer or flow resistance for solutions up to this order. The relative magnitude of centrifugal buoyancy to gravitational buoyancy is proportional to the acceleration ratio $H\Omega^2/g$. Consequently, at high values of this parameter, centrifugal buoyancy is expected to be dominant.

Although this investigation was largely stimulated by problems associated with cooling rotating components, where entry length effects are of considerable importance, the analysis presented has been restricted to the determination of asymptotic profiles and their influence on flow resistance and heat transfer.

The author would like to express his appreciation to Dr J. E. Wilkinson, Department of Applied Mathematics, University College of Swansea, and Mr T. H. Davies, Department of Mechanical Engineering, University College of Swansea, for their valuable discussions during the preparation of this paper.

Appendix

Numerical coefficients in the second-order solutions for the velocity and temperature distributions

ψ_2 *coefficients*

b	C_{2b}	D_{2b}
1	1·2040	3·7564
2	−2·0617	−9·1607
3	0·1500	7·5000
4	1·1000	−2·6000
5	−0·4250	0·5625
6	0·0343	−0·0600
7	−0·0016	0·0018

c	C_{2c+1}	D_{2c+1}
0	1·0327	3·3053
1	−2·4926	−8·1721
2	1·9844	6·9010
3	−0·6380	−2·6042
4	0·1302	0·6771
5	−0·0182	−0·1172
6	0·0015	0·0104
7	−0·0000	−0·0003

d	C_{2d+1}
0	0·0436
1	−0·1130
2	0·1042
3	−0·0451
4	0·0117
5	−0·0015
6	0·0001

e	C_{2e+1}
0	2986
1	−6356
2	3800
3	−450
4	30
5	−1

W_2 *coefficients*

b	E_b
0	−0·6148
1	3·0625
2	−6·3406
3	7·0583
4	−4·5609
5	1·7125
6	−0·3448
7	0·0286
8	−0·0008

c	E_{2c}	F_{2c}
1	−0·1152	−0·5135
2	0·3409	1·2521
3	−0·4327	−1·1451
4	0·3112	0·5000
5	−0·1313	−0·1083
6	0·0295	0·0161
7	−0·0024	−0·0013
8	0·0000	0·0000

i	E_{2i+1}
0	−0·0025
1	0·0055
2	−0·0047
3	0·0022
4	−0·0006
5	0·0001
6	−0·0000
7	0·0000

d	E_{2d+1}	F_{2d+1}
0	−430·5304	− 79·1716
1	944·0493	158·9949
2	−781·9909	−119·9962
3	329·7889	51·9983
4	−68·6998	−13·4996
5	7·6800	1·6799
6	−0·2971	−0·0057

e	E_{2e+1}	F_{2e+1}
0	−0·0595	−0·1888
1	0·1291	0·4132
2	−0·1039	−0·3405
3	0·0413	0·1438
4	−0·0080	−0·0326
5	0·0011	0·0056
6	−0·0001	−0·0007
7	0·0000	0·0000
8	−0·0000	−0·0000

j	E_{2j}
0	−36·5100
1	52·7500
2	−19.0000
3	3·0000
4	−0·2500
5	0·0100

η_2 coefficients

b	J_{2b}	K_{2b}	L_{2b}
0	−0·0635	−0·0409	−0·1340
1	0·1537	0·1985	0·6901
2	−0·1914	−0·3998	1·5059
3	0·1761	0·4342	1·8212
4	−0·1103	−0·2793	−1·3434
5	0·0456	0·1116	0·6242
6	−0·0119	−0·0281	−0·1789
7	0·0018	0·0041	0·0287
8	−0·0001	−0·0003	−0·0021
9	0·0000	0·0000	0·0001

c	J_{2c}	K_{2c}	L_{2c}
1	0·0036	0·0523	0·1285
2	−0·0096	−0·1318	−0·3508
3	0·0107	0·1279	0·4765
4	−0·0072	−0·0589	−0·3693
5	0·0032	0·0099	0·1739
6	−0·0009	0·0009	−0·0500
7	0·0002	−0·0003	−0·0082
8	−0·0000	0·0000	−0·0006
9	0·0000	−0·0000	0·0000

d	J_{2d+1}	K_{2d+1}
0	−60·1647	−44·5543
1	148·8163	99·4565
2	−135·8354	−85·2048
3	61·4165	37·4124
4	−16·4474	−7·7000
5	2·3975	0·6225
6	−0·1886	−0·0329
7	0·0058	0·0006

e	J_{2e+1}	K_{2e+1}	L_{2e+1}
0	−0·0073	−0·00177	−0·0821
1	0·0143	0·0762	0·1983
2	−0·0103	−0·0932	−0·1965
3	0·0042	0·0459	0·1099
4	−0·0010	−0·0132	−0·0363
5	0·0001	0·0023	0·0079
6	−0·0000	−0·0003	−0·0013
7	0·0000	0·0000	0·0001
8	−0·0000	−0·0000	−0·0000
9	0·0000	0·0000	0·0000

i	J_{2i+1}	K_{2i+1}
0	−0·0003	−0·1026
1	0·0015	0·1090
2	−0·0018	−0·0112
3	0·0008	0·0067
4	−0·0002	−0·0024
5	0·0000	0·0006
6	−0·0000	−0·0001
7	0·0000	0·0000
8	−0·0000	−0·0000

j	J_{2j}
0	6·3139
1	−9·1275
2	3·2969
3	−0·5278
4	0·0469
5	−0·0025
6	0·0001

REFERENCES

ALPERT, S., GREY, R. E. & FLASHAR, W. O. 1960 *Trans. A.S.M.E.*

BARUA, S. N. 1954 *Proc. Roy. Soc.* A, **227**, 133.

GOLDSTEIN, S. 1957 *Modern Developments in Fluid Dynamics*, Vol. 2. Oxford University Press.

HOLZWORTH, H. 1938 Die Entwicklung der Halzworth-Gas Turbine, Holzworth-Gas turbinen, G.m.b.H. Muehlheim-Ruhr.

LESLIE, F. M. 1960 *J. Fluid Mech.* **7**, 115.

LIGHTHILL, M. J. 1949 *Phil. Mag.*, **40**, 1179.

LIGHTHILL, M. J. 1953 *Quart. J. Mech. Appl. Phys.* **6**, 398.

MARTIN, B. W. 1955 *Proc. Roy. Soc.* A, **230**, 502.

MARTIN, B. W. & COHEN, H. 1954 *Brit. J. Appl. Phys.* **5**, 91.

MORTON, B. R. 1959 *Quart. J. Mech. Appl. Math.* **12**, 410.

ROSSBY, C. G. 1949 *Tellus*, **1**, 54.